The Agulhas - A Logarithmic Western Boundary Current

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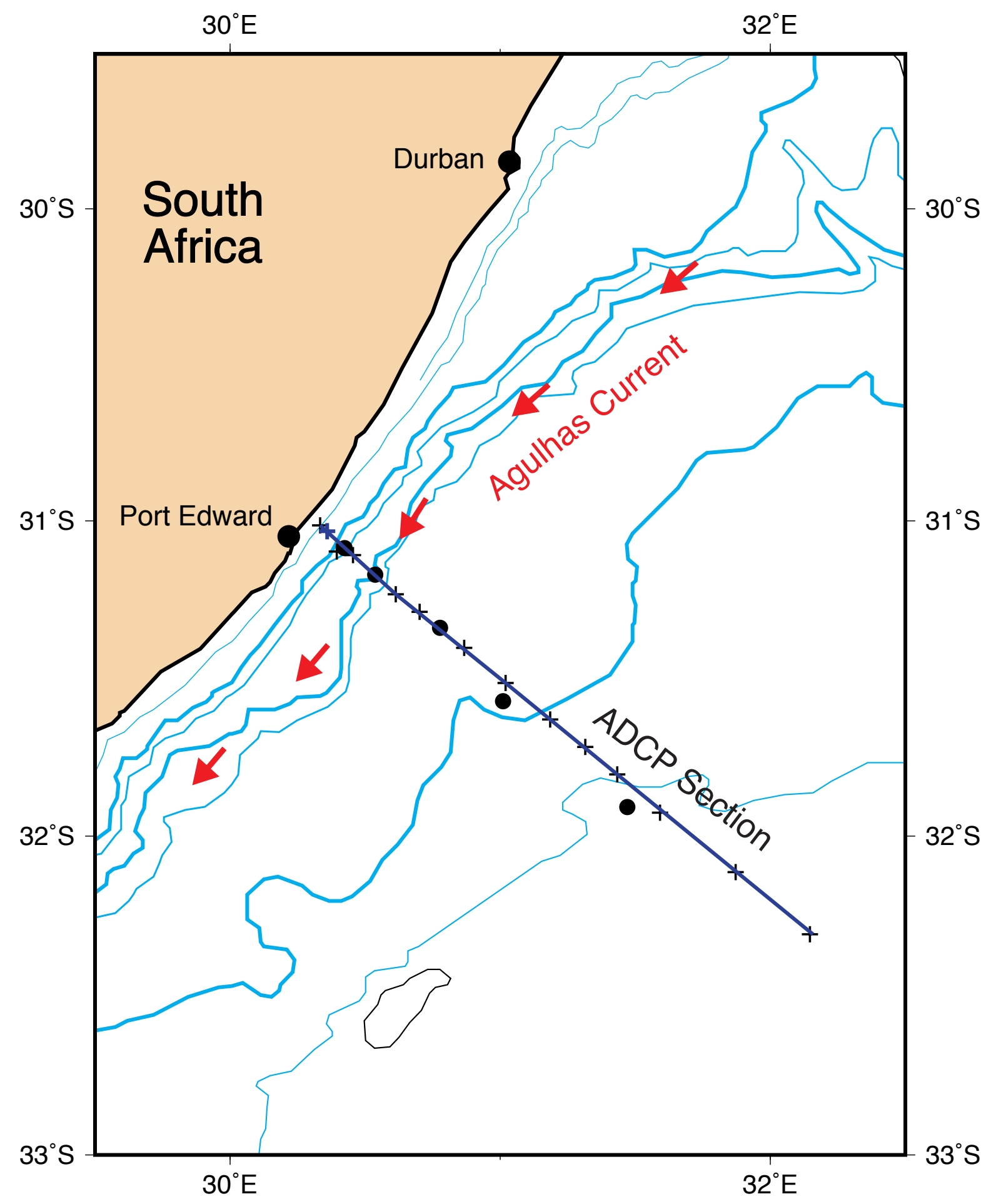


Figure 1.. The position of the ADCP section made across the Agulhas Current

The Agulhas Current

In 1987, Harry Bryden and colleagues used an Acoustic Doppler Current Profiler (ADCP) to measure the velocity structure of the Agulhas Current along a section 200 km south-west of Durban. The data from 237m, the greatest depth for which continuous data was obtained is shown in figure 2. It shows a narrow maximum about 13 km from the continental slope and decays exponentially offshore.

Bottom friction is unlikely to be involved because the depths are too great (1440m at the current maximum) and there is evidence for a counter current at 1000m. If a Munk solution assuming constant horizontal viscosity is fitted to all of the data, the current maximum is too far offshore. If the Munk solution is fitted to the current maximum region the solution decays too rapidly. Thus another theory is needed.

One possibility is that the viscosity initially increases linearly away from the coast. This possibility is investigated here.

Western Boundary Current I

In a steady western boundary current, along a north-south boundary, the dominant terms in the eastward and northward momentum equations are:

$$-fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x},$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left(A(x)\frac{\partial}{\partial x}\right)v.$$

x and y are the east and north coordinates, f is the Coriolis term, v the northward velocity, ρ is density, p pressure and A(x) the viscosity.

If the viscosity term increases linearly away from the coast (due possibly to the increasing scale of the horizontal turbulence) then A(x) equals A*x. Eliminating p by taking the curl of the equation,

$$\frac{\partial^2}{\partial x^2}\left(x\frac{\partial}{\partial x}\right)v(x) = k_0^2 v(x),$$

where k_0^2 equals β/A and β equals ∂f/∂y. To obtain analytic solutions, first transform to non-dimensional form, where x equals $(A/\beta)^{1/2}z$,

$$\frac{\partial^2}{\partial z^2}\left(z\frac{\partial}{\partial z}\right)v(z) - v(z) = 0.$$

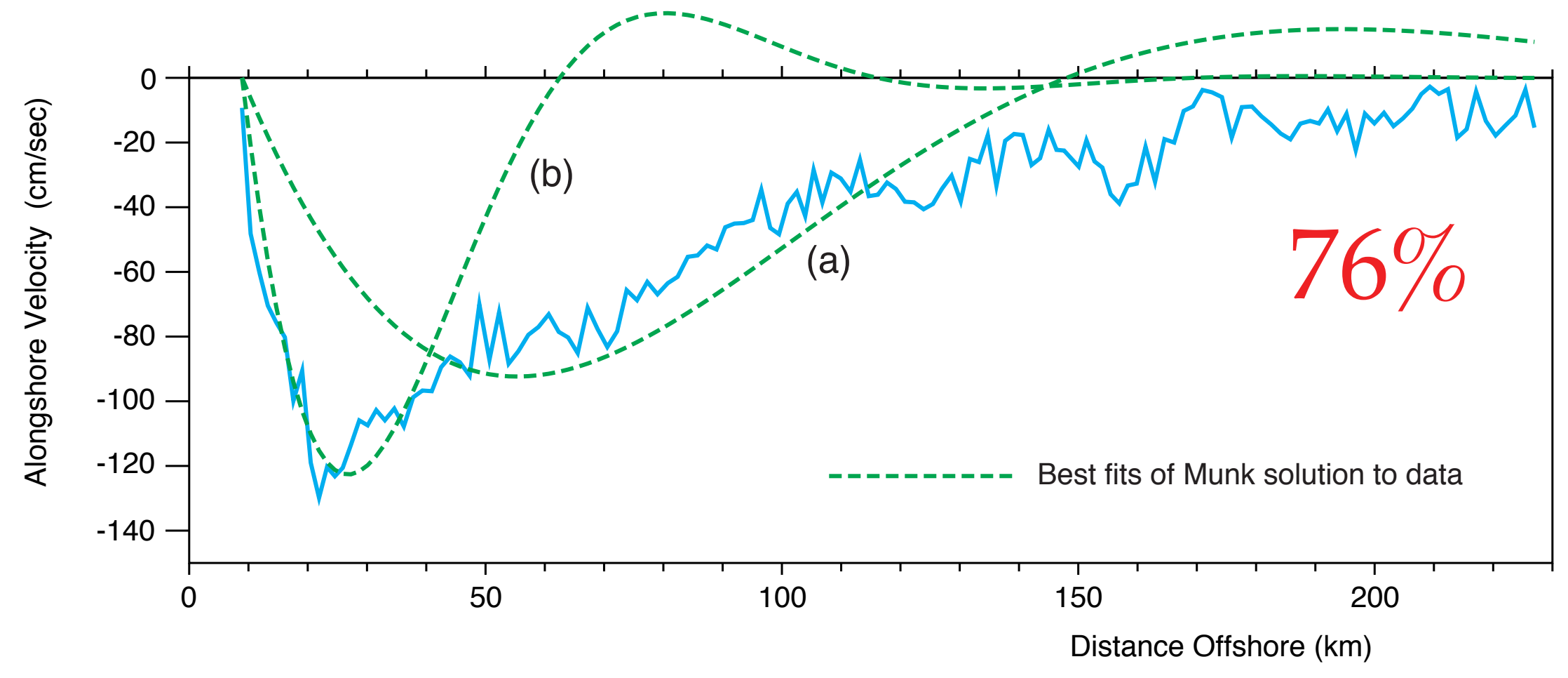


Figure 2. The observed current profile (blue) and the solutions obtained by fitting the Munk solution to (a) the full data set and (b) the peak region only. Curve (a) fits only 76% of the variance in the data.

The Analytic Functions

$$\frac{\partial^2}{\partial z^2}\left(z\frac{\partial}{\partial z}\right)v(z) - v(z) = 0.$$

This non-dimensional form of the boundary current equation has three independent solutions. By writing the solutions as a power series, and substituting, these are found to be:

$$v_0(z) = 1 + \frac{z^2}{1\times 2^2} + \frac{z^4}{1\times 2^2\times 3\times 4^2} + \ldots,$$

$$= \sum_{n=0}^{\infty} z^{2n}\Big/\left[2^n\,\Gamma(2n+1)\Gamma(n+1)\right],$$

$$v_1(z) = z + \frac{z^3}{1^2\times 2\times 3^2} + \frac{z^5}{1^2\times 2\times 3^2\times 4\times 5^2} + \ldots,$$

$$= \sum_{n=0}^{\infty} z^{2n+1}\left[2^n\,\Gamma(n+1)/\Gamma(2n+2)^2\right],$$

$$v_2(z) = \ln(z)v_0(z) + \left[\frac{z^2}{1\times 2^2}\left(-\frac{1}{1}-\frac{2}{2}\right) + \frac{z^4}{1\times 2^2\times 3\times 4^2}\left(-\frac{1}{1}-\frac{2}{2}-\frac{1}{3}-\frac{2}{4}\right)+\ldots\right].$$

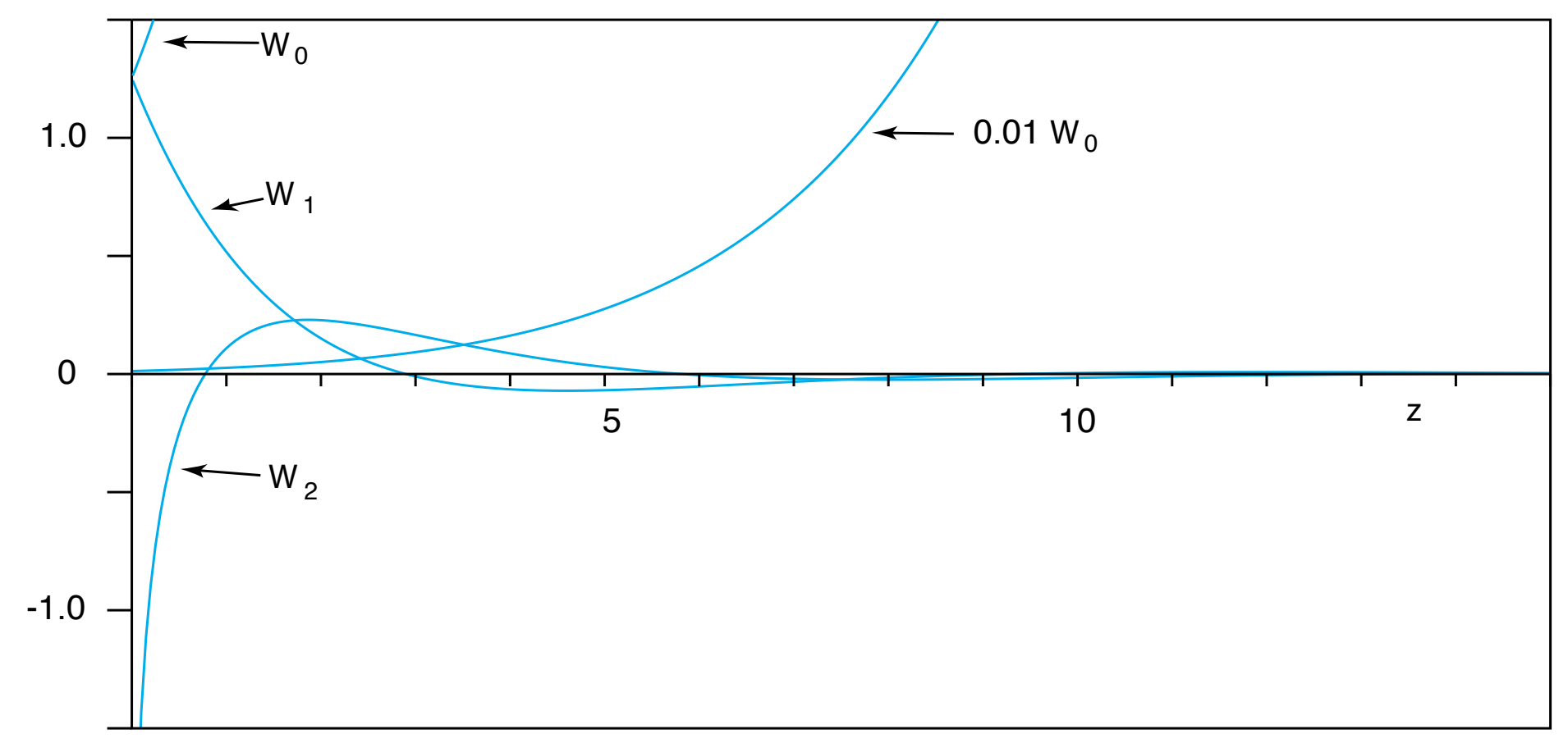
All three tend to infinity at large z. However if we combine them so,

$$W_0(z) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} v_0(z) + v_1(z),$$

$$W_1(z) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} v_0(z) - v_1(z),$$

$$W_2(z) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(v_2(z) + \left(\frac{3}{2}\gamma - \frac{1}{2}\ln(2)\right)v_0(z)\right),$$

where γ is Euler's constant (0.577215664 ...) then W_0 tends to infinity and W_1 and W_2 tend to zero at large z. These functions are plotted below.



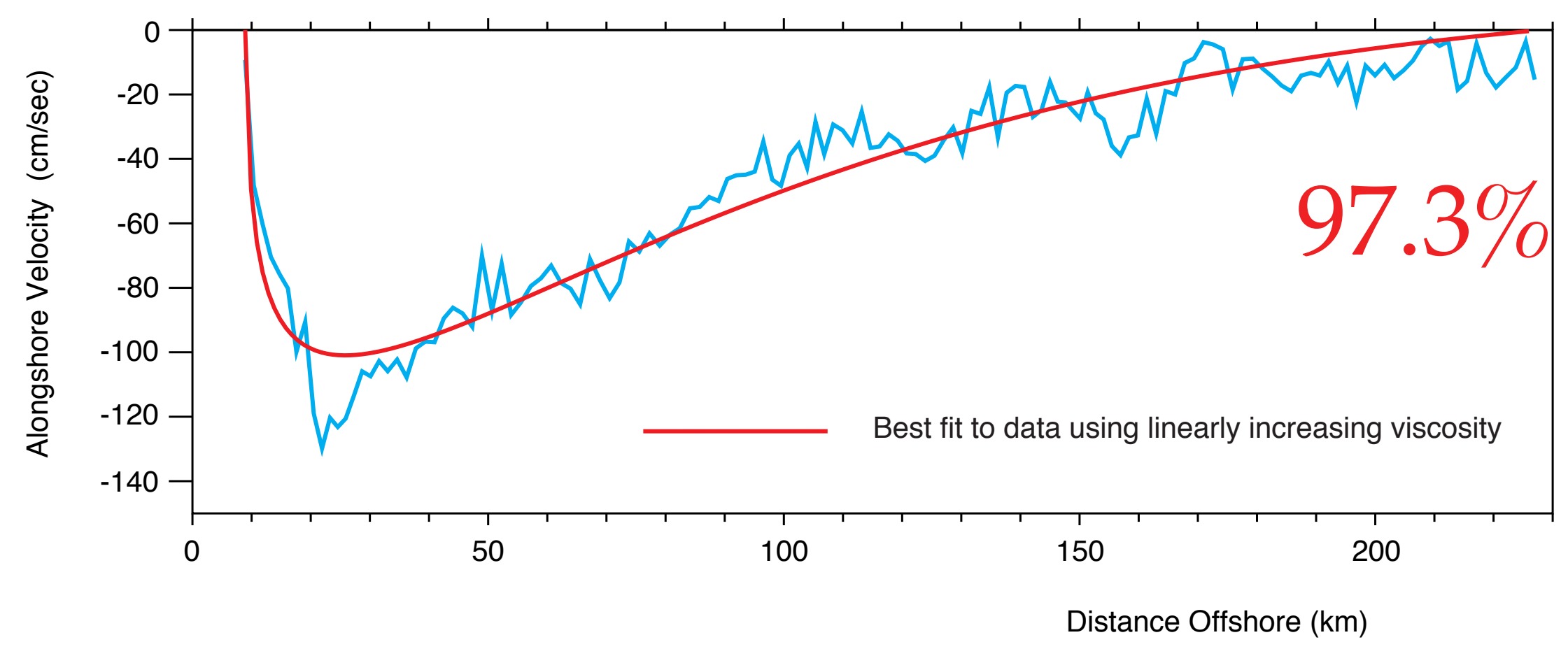


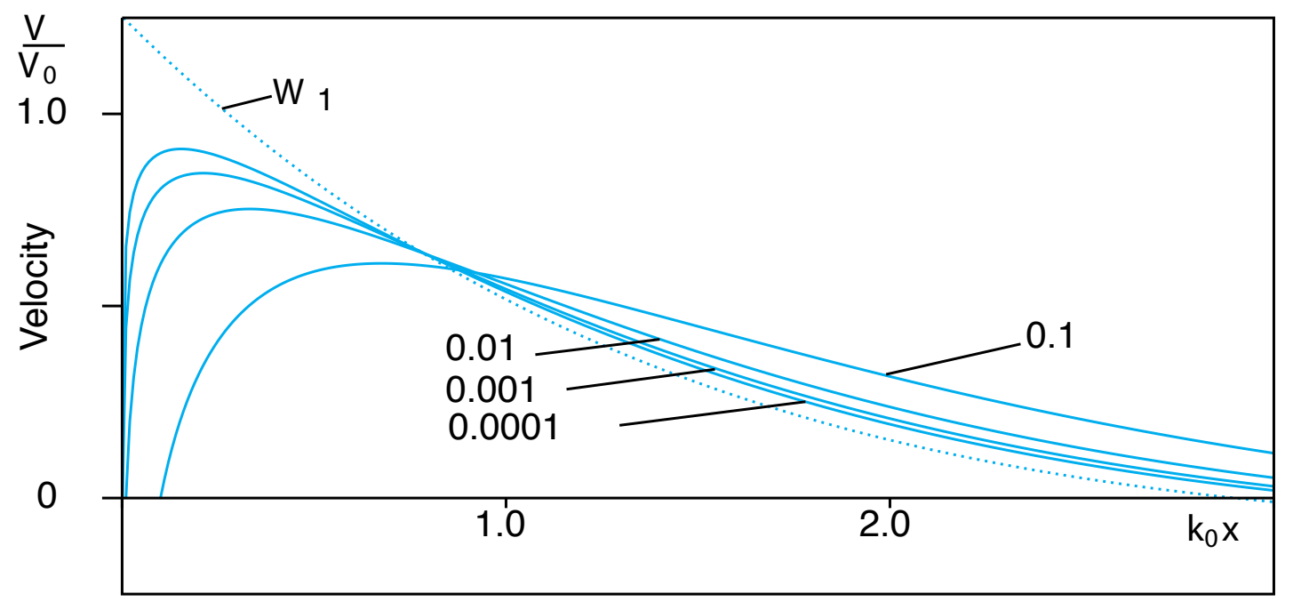
Figure 3. The best fit to the data using the new set of boundary current functions. The new curve explaines 97.3% of the variance in the data.

Western Boundary Current II

The full solution can be constructed from a linear combination of the analytic functions. At large distances from the coast, the velocity should tend to zero so the functions W_0 cannot be involved. Near the coast the function W_2 has a logarithmic singularity. We therefore set the solution to zero at a distance ε from the coast, where ε the roughness length, represents the effect of small roughness elements near the coast.

$$AW_1(\varepsilon) + BW_2(\varepsilon) = 0.$$

The solutions for various values of the roughness length with unit total transport are shown below.



Let us assume that the Agulhas Current data can be fitted by a function of the form,

$$v(x) = A\,W_1\left(\frac{x}{x_0}\right) + B\,W_2\left(\frac{x}{x_0}\right).$$

The solution has three free parameters, the horizontal length scale x_0 (equal to k_0^{-1}), the roughness length ε and the total transport. These were estimated by minimising the variance between the analytic solution and the observed data.

The resulting best fit is shown in figure 3. Although the new fit has only one more parameter than the Munk solution it produces a much better fit to both the position and magnitude of the peak of the current and to its offshore tail.

Stress, Force *and moment*

Let the viscous force F(x) acting on a boundary current be ∂/∂x S(x), where S(x), the shear stress, is continuous but otherwise of general form. The momentum equations are then,

$$-\rho_0 fv = -\frac{\partial p}{\partial x}, \qquad 0 = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}S(x).$$

Eliminating the pressure,

$$\frac{\partial^2}{\partial x^2}S(x) = \rho_0\beta v(x).$$

Let the transport offshore of x be T(x). Then using the above equation,

$$T(x) = \int_x^{\infty} v(x')\,\partial x' = \frac{1}{\rho_0\beta}\int_x^{\infty}\left(\frac{\partial^2}{\partial x'^2}S(x')\right)\partial x' = \frac{1}{\rho_0\beta}\left[F(\infty) - F(x)\right].$$

The current decays offshore, so at infinity the force F(∞) is zero. Thus,

$$F(x) = -\rho_0\beta\,T(x). \qquad F(0) = -\rho_0\beta\,T(0).$$

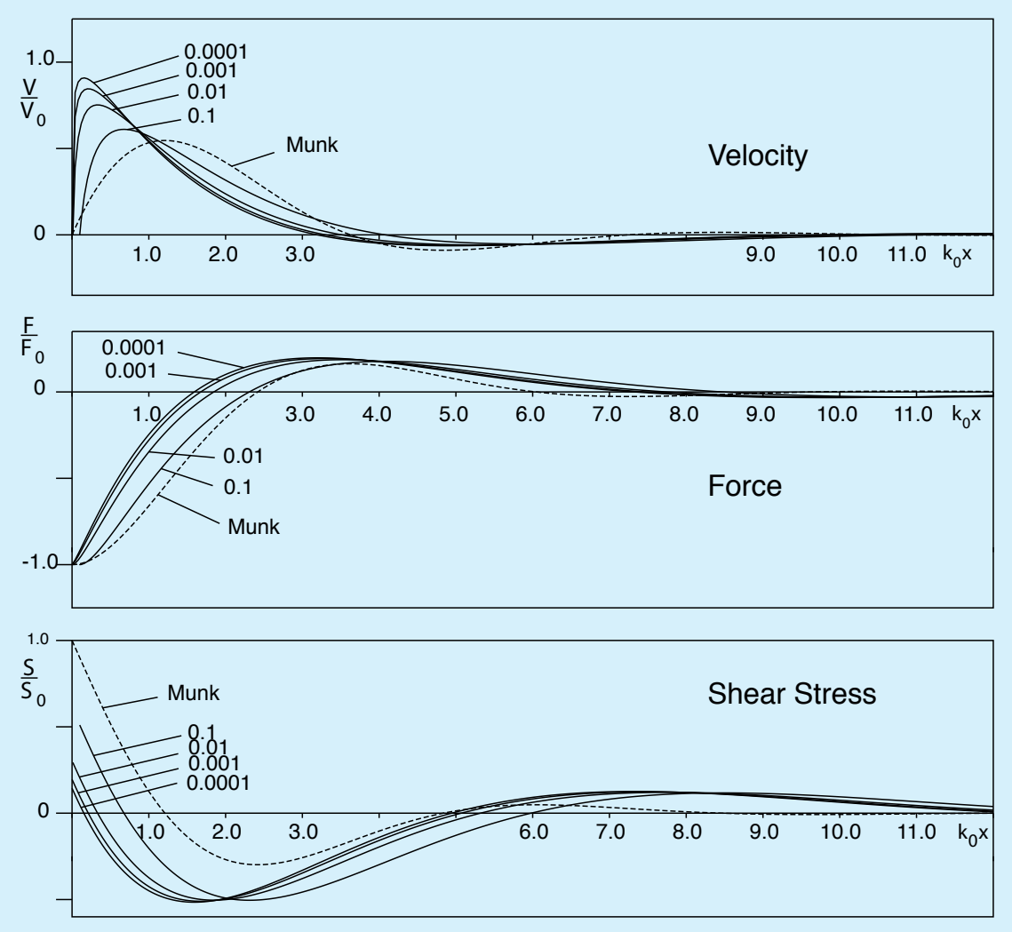
The force is balanced by the pressure gradient parallel to the shore. At the coast both the alongshore pressure gradient and the sea level slope along the coast, are proportional to the total transport in the boundary current.

In the same way it can be shown that the shear stress is proportional to the first moment of the offshore current field. Thus the stress on the coast depends on the first moment of the full current.

$$S(x) = -\int_x^{\infty} F(x')\,dx' = \rho_0\beta\int_x^{\infty}(x' - x)v(x')\,dx'.$$

$$S(0) = \rho_0\beta\int_0^{\infty} xv(x)dx.$$

These effects are illustated below using a series of boundary currents with the same total transport. These consists of the Munk constant viscosity solution plus a group of linear viscosity solutions with different roughness lengths. As the current becomes narrower, the stress on the coast is reduced but the net force acting on water adjacent to the coast remains constant.



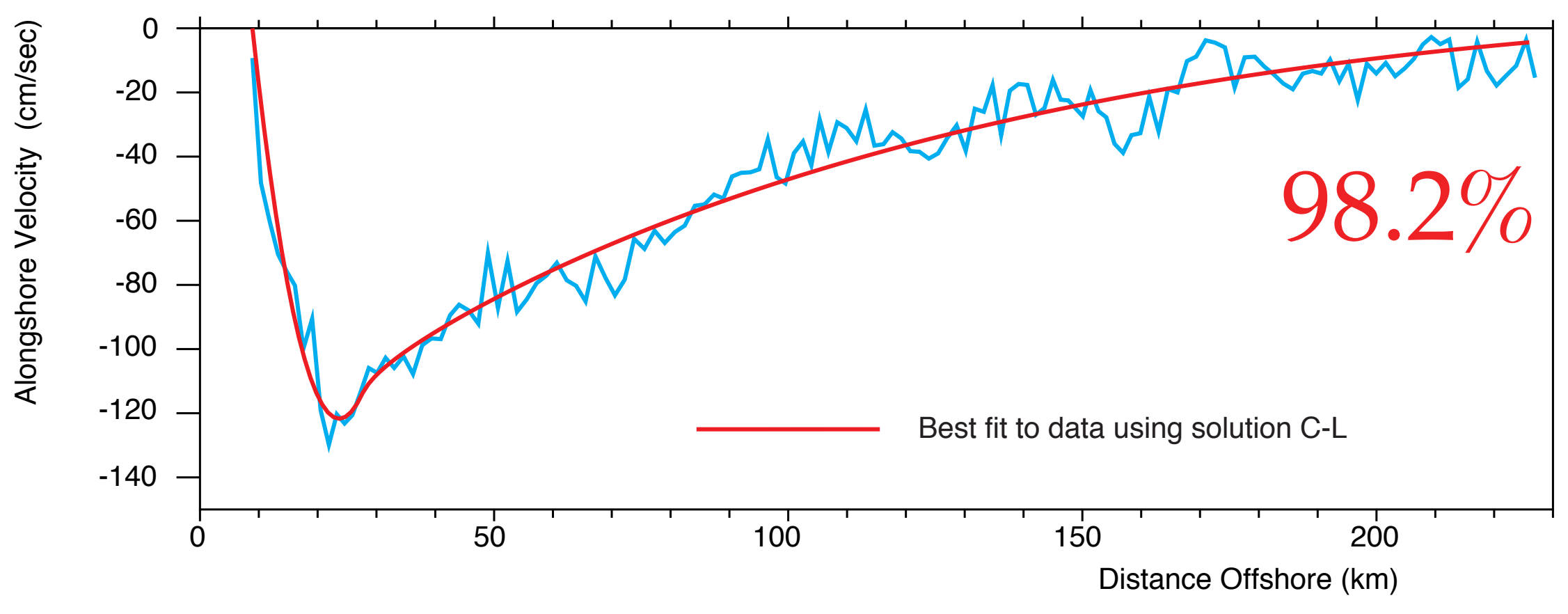


Figure 4. The best fit to the data using an additional constant viscosity region inshore of the peak of the current. This solution explaines 98.2% of the variance in the data. Just one extra parameter has reduced the residual variance by almost a third.

Constant Viscosity Layer

There remain three noticeable differences between the new solution and the Agulhas Current data. In the analytic solution the inshore current shear is too large, the current maximum is too wide and there is an offshore zero crossing not present in the data.

To progress further, we consider the possibilities that offshore the viscosity eventually reaches some maximum value and that near the coast there is a similar minimum value. Such a minimum value may occur if momentum transfer between the coast and the current maximum is dominated by a single size range of eddies. Solutions in regions of constant viscosity were given by Munk,

$$v_{m1}(x) = k_m\frac{2}{\sqrt{3}}\exp\left(-\frac{1}{2}k_m x\right)\sin\left(\frac{\sqrt{3}}{2}k_m x\right),$$

$$v_{m2}(x) = k_m\,2\exp\left(-\frac{1}{2}k_m x\right)\cos\left(\frac{\sqrt{3}}{2}k_m x\right),$$

$$v_{m3}(x) = k_m\exp\left(+\frac{1}{2}k_m x\right).$$

The boundary conditions between neighbouring regions are obtained by integrating the basic equation between x-ε and x+ε, where x is the boundary and ε is infinitesimally small. In physical terms, the shear stress, force and velocity should all to be continuous across the boundary.

$$\frac{\partial}{\partial x}\left(A(x)\frac{\partial}{\partial x}\right)v(x)\bigg|_{x_1-\varepsilon} = \frac{\partial}{\partial x}\left(A(x)\frac{\partial}{\partial x}\right)v(x)\bigg|_{x_1+\varepsilon},$$

$$\left(A(x)\frac{\partial}{\partial x}\right)v(x)\bigg|_{x_1-\varepsilon} = \left(A(x)\frac{\partial}{\partial x}\right)v(x)\bigg|_{x_1+\varepsilon},$$

$$v(x)\big|_{x_1-\varepsilon} = v(x)\big|_{x_1+\varepsilon}.$$

Three cases are considered. In the first C-L, viscosity is constant near the shore and increases linearly offshore. The second L-C, is linear near the coast and reaches a maximum offshore. The third C-L-C has constant regions at the coast and offshore and a linear region in between. In all three cases the viscosity is continuous with no sudden jumps. The minimum variances for these solutions, the pure linear viscosity case L, discussed on the left and the pure Munk solution C are given in the following table.

Model	Residual Variance cm^2s^{-2}	Variance explained %	x_{m1} km	x_1 km	x_w km	x_2 km
Data	449080.3					
C	107611.2	76.04	38.35			
L	12055.7	97.32		0.23	64.57	
C-L	8104.6	98.20	18.99	18.61	81.74	
L-C	12056.6	97.32		0.22	64.65	581.8
C-L-C	8104.6	98.20	18.99	18.62	81.74	876.1

x_1 and x_2 are the positions of the inner and outer boundaries between the regions. x_{m1} and x_w are the scale lengths of the inner and linearly varying regions respectively. For model L where there is no coastal constant viscosity region, x_1 is the roughness length.

The only significant improvement, over case L, is that of case C-L which reduces the residual variance by a third at the cost of only one additional parameter. In the other two cases the improvement in fit is not statistically significant.

This solution for case C-L, figure 4, gives a much better fit to the observations inshore of the current maximum and also near the maximum itself.

Webb, D.J. 1999: An analytic model of the Agulhas Current as a western boundary current with linearly varying viscosity. Journal of Physical Oceanography, 29, 1517-1527.