Improved Advection Schemes for Ocean Models

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The Problem

The standard central-difference advection scheme used in the Bryan and Cox ocean model suffers from a large amount of computational noise. The QUICK advection scheme overcomes the noise problem but in its normal form it cannot be used with the leapfrog/Euler-forward timestepping scheme used by the ocean model. An alternative timestepping scheme like predictor-corrector can be used but this is computationally expensive to run.

Here we show that the problem can be overcome by splitting the QUICK operator into a $O(\delta x^2)$ advective scheme and a velocity dependent biharmonic diffusion term. These can be timestepped using the leapfrog and Euler-forward scheme respectively leading to a significant increase in model efficiency.

Central Differences

In the ocean model, the advection-diffusion equation for tracers is,

$$\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla_h)T + w\frac{\partial T}{\partial z} = A_h \nabla_h^2 T + \kappa \frac{\partial^2 T}{\partial T^2}.$$

T is the tracer, t time, z depth, $\mathbf{u}$ the horizontal velocity, w the vertical velocity and ∇_h the horizontal gradient operator. A_h is the horizontal and κ the vertical diffusion coefficient.

We are interested in the advection of small scale features of the tracer field by the large scale velocity field. This is easiest to analyse if we study the properties of the different advection schemes in a one-dimensional channel with constant advective velocity. If U is the advective velocity and x the distance along the channel, then the advection equation is,

$$\frac{\partial T(x,t)}{\partial t} = -U\frac{\partial T(x,t)}{\partial x}.$$

Assume that the channel is split into cells of length δx. x_i is the position of the i'th cell and T_i the value of tracer in the cell. If $T_{i+1/2}$ is the value on the interface between cells i and $i+1$, then

$$\frac{\partial T_i}{dt} = -U\frac{(T_{i+1/2} - T_{i-1/2})}{\delta x}.$$

The standard central difference scheme used by the ocean model approximates $T_{i+1/2}$ by,

$$T_{i+1/2} = (T_{i+1} + T_i)/2.$$

and so,

$$\frac{\partial T_i}{dt} = -U\frac{(T_{i+1} - T_{i-1})}{2\delta x}$$

The error introduced by the scheme can be determined by substituting in the Taylor expansion,

$$T(x+a) = \exp\left(a\frac{\partial}{\partial x}\right)T(x),$$

giving,

$$\frac{\partial T}{\partial t} = -U\left(\frac{\partial T}{\partial x} + \frac{\delta x^2}{3!}\frac{\partial^3 T}{\partial x^3} + \frac{\delta x^4}{5!}\frac{\partial^5 T}{\partial x^5} + \dots\right).$$

Comparison with the original advection equation shows that the leading error is $O(\delta x^2)$. The effect on individual Fourier components can be seen by substituting $T(x,t)$,

$$T(x,t) = \exp(ikx - i\omega t),$$

From the full advection equation, the dispersion relation should be,

$$\omega = kU.$$

For the central difference scheme it is,

$$\omega = kU\frac{\sin(k\delta x)}{k\delta x}.$$

These two functions are plotted in fig 1. At short wavelengths the errors of the central-difference scheme are large. The group velocity $\partial\omega/\partial k$ has the wrong sign so the energy at short wavelengths is advected upstream against the current. This produces a large amount of short wavelength noise in the solution (see box on the right).

The QUICK Scheme

Leonard's QUICK scheme has been shown to overcome this problem of central-differences. It uses a three point upstream formula for the tracer value at box boundaries. If U is positive then for a regular grid,

$$T_{i+1/2} = \frac{1}{2}(T_{i+1} + T_i) - \frac{1}{8}(T_{i+1} - 2T_i + T_{i-1}).$$

so,

$$\frac{\partial T_i}{\partial t} = -U\left(\frac{T_{i+1} - T_{i-1}}{2\delta x} - \frac{T_{i+1} - 3T_i + 3T_{i-1} - T_{i-2}}{8\delta x}\right).$$

For the QUICK scheme, the Taylor expansion and dispersion relation are,

$$\frac{\partial T}{\partial t} = -U\left(\frac{\partial T}{\partial x} + \frac{1}{4}\frac{\delta x^2}{3!}\frac{\partial^3 T}{\partial x^3} + \frac{3}{2}\frac{\delta x^3}{4!}\frac{\partial^4 T}{\partial x^4} + \dots\right),$$

$$\omega = kU\left[\frac{\sin(k\delta x)}{k\delta x} + \frac{1}{8}\left(\frac{2\sin(k\delta x) - \sin(2k\delta x)}{k\delta x}\right) - i\left(\frac{3 + \cos(2k\delta x) - 4\cos(k\delta x)}{8k\delta x}\right)\right].$$

Note that the dispersion relation now includes an imaginary component which damps out short waves. The leading error in the Taylor's expansion is smaller than before and as a result, at long wavelengths the dispersion relation is improved. At short wavelengths the relation is worse but these are just the wavelengths which are damped out by the scheme. The overall result is a significant improvement over central differences.

However in the Bryan and Cox ocean model the scheme cannot be used with the standard leapfrog/Euler-forward timestepping scheme. This is because QUICK includes both advective and diffusive terms. Leapfrog is unstable with diffusive terms and Euler forward is unstable with advective terms. Instead a predictor-corrector scheme has to be used. This doubles the computational cost.

For further details see: Webb, D.J., de Cuevas, B.A. and Richmond, C.S., 1998: Improved Advection Schemes for ocean Models. Journal of Atmospheric and Oceanic technology, 15, 1173-1187.

Dispersion Curves

The figures show the real and imaginary parts of the dispersion curves for the Central Difference, QUICK, SQ and MSQ schemes. The speed at which energy propagates, i.e. the group velocity, is given by the slope of the dispersion relation ($\partial\omega/dk$).

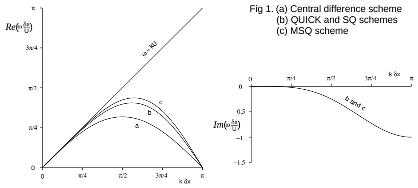
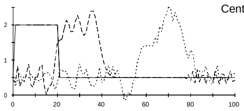


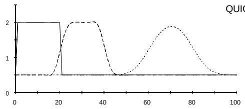
Fig 1. (a) Central difference scheme
(b) QUICK and SQ schemes
(c) MSQ scheme

Tests using a 1-D Channel Model

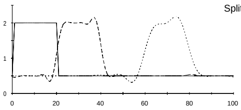
The figures show the result of tests of the different schemes in a cyclic 1-D channel. The channel is of length 100, the grid spacing and timestep both equal 1 and the velocity equals 0.2. The solid line shows the initial tracer distribution, the dashed line shows the distribution after 100 timesteps and the dotted line the distribution after 800 timesteps. The QUICK solutions are more damped than the SQ ones because of the extra numerical viscosity introduced by the predictor-corrector timestepping scheme.



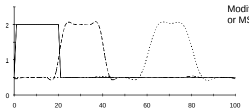
Central Differences



QUICK



Split-QUICK or SQ

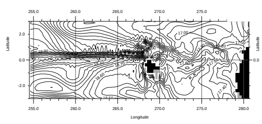


Modified Split-QUICK or MSQ

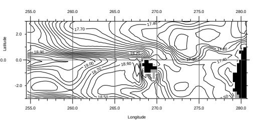
Tests using the OCCAM Global Model

A series of tests were carried out using the1/4 degree version of the OCCAM global ocean model. Each test lasted for 60 days and started from the model state at the beginning of year eight of the main model run. The figures on the right is show the surface temperature fields in the Agulhas Retroflection Region at the end of the 60 day period obtained by applying the Central Difference, Split-QUICK and Modified Split-QUICK schemes to both the tracer and velocity fields. The figures below show the temperature and current speed at a depth of 200m in the eastern equatorial Pacific also at the end of the 60 day period. In this case the Split-QUICK scheme was applied only to the tracer field.

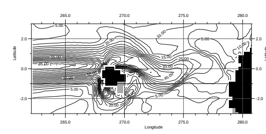
Temperature at 200 m Centr al Differences



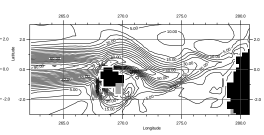
Temperature at 200 m Split-Q UICK



Speed at 200 m Centr al Differences



Speed at 200 m Split-Q UICK



Split-QUICK or SQ

The key point of this poster is that it is possible to partition the QUICK operator into its advective and diffusive parts. This can be done for both constant and variable grid spacings. With a constant grid the basic interpolation equation becomes,

$$T_{i+1/2} = \left(\frac{T_{i+1} + T_i}{2} - \frac{T_{i+2} - T_{i+1} - T_i + T_{i-1}}{16}\right) + \frac{|U|}{U}\left(\frac{T_{i+2} - 3T_{i+1} + 3T_i - T_{i-1}}{16}\right).$$

Substitution into the advection equation gives,

$$\frac{\partial T_i}{\partial t} = -U\left(\frac{T_{i+1} - T_{i-1}}{2\delta x} - \frac{T_{i+2} - 2T_{i+1} + 2T_{i-1} - T_{i-2}}{16\delta x}\right) - |U|\left(\frac{T_{i+2} - 4T_{i+1} + 6T_i - 4T_{i-1} + T_{i-2}}{16\delta x}\right).$$

The first term is the advective and the second the diffusive term. The first term can be timestepped using leapfrog, the second using Euler-forward. Thus in this form the scheme introduces little additional computational cost. The Taylor series and the dispersion relation are the same as for the QUICK scheme.

$$\frac{\partial T}{\partial t} = -U\left(\frac{\partial T}{\partial x} + \frac{1}{4}\frac{\delta x^2}{3!}\frac{\partial^3 T}{\partial x^3} + \dots\right) - |U|\left(\frac{3}{2}\frac{\delta x^3}{4!}\frac{\partial^4 T}{\partial x^4} + \dots\right).$$

$$\omega = kU\left[\left(\frac{\sin(k\delta x)}{k\delta x} + \frac{2\sin(k\delta x) - \sin(2k\delta x)}{8k\delta x}\right) - i\frac{|U|}{U}\left(\frac{3 + \cos(2k\delta x) - 4\cos(k\delta x)}{8k\delta x}\right)\right].$$

Modified Split-QUICK or MSQ

The QUICK scheme reduces the size of the $O(\delta x^2)$ term in the Taylor series expansion but does not remove it completely. By changing the operator slightly one can obtain an even better scheme at no additional cost. The basic interpolation and advection equations are now,

$$T_{i+1/2} = \left(\frac{T_{i+1} + T_i}{2} - \frac{T_{i+2} - T_{i+1} - T_i + T_{i-1}}{12}\right) + \gamma\frac{|U|}{U}\left(\frac{T_{i+2} - 3T_{i+1} + 3T_i - T_{i-1}}{16}\right).$$

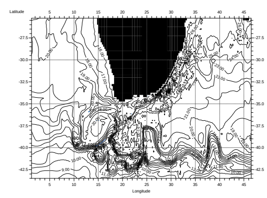
$$\frac{\partial T_i}{\partial t} = -U\left[\left(\frac{T_{i+1} - T_{i-1}}{2\delta x}\right) - \left(\frac{T_{i+2} - 2T_{i+1} + 2T_{i-1} - T_{i-2}}{12\delta x}\right)\right] - \gamma|U|\left[\frac{T_{i+2} - 4T_{i+1} + 6T_i - 4T_{i-1} + T_{i-2}}{16\delta x}\right].$$

The Taylor series and dispersion equations become,

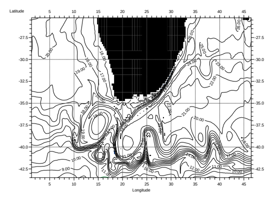
$$\frac{\partial T}{\partial t} = -U\left[\frac{\partial T}{\partial x} - 4\frac{\delta x^4}{5!}\frac{\partial^5 T}{\partial x^5} + \dots\right] - \gamma|U|\left[\frac{3}{2}\frac{\delta x^3}{4!}\frac{\partial^4 T}{\partial x^4} + \frac{15}{2}\frac{\delta x^5}{6!}\frac{\partial^6 T}{\partial x^6} + \dots\right].$$

$$\omega = kU\left[\left(\frac{4}{3}\frac{\sin(k\delta x)}{k\delta x} - \frac{1}{6}\frac{\sin(2k\delta x)}{k\delta x}\right) - i\gamma\frac{U}{|U|}\left(\frac{3 + \cos(2k\delta x) - 4\cos(k\delta x)}{8k\delta x}\right)\right].$$

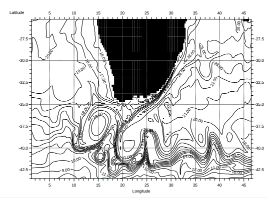
The leading error term is now the $O(\delta x^3)$ diffusion term. Otherwise advection is correct to $O(\delta x^4)$. For the damping term, a value of γ near 1 was found to give the best results.



Surface Temperature
Central Differences



Surface Temperature
Split-QUICK



Surface Temperature
Modified Split-QUICK

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